ERRATA FOR *MATHEMATICAL GAUGE THEORY. WITH APPLICATIONS TO THE STANDARD MODEL OF PARTICLE PHYSICS* (SPRINGER 2017)

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Page numbers and line numbers refer to the published version.

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Please send additional corrections by e-mail to
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45 (15)	An analogous result holds for the Lie algebra $\mathfrak{gl}(n,\mathbb{K})$ of $\mathrm{GL}(n,\mathbb{K})$ for $\mathbb{K}=\mathbb{C},\mathbb{H}$, i.e. $\mathfrak{gl}(n,\mathbb{K})=\mathrm{Mat}(n\times n,\mathbb{K})$ as a real vector space and the Lie bracket is given by the standard commutator of matrices.
45 (-4, -2)	replace e by the unit matrix I
46 (12-13)	Here (and in calculations elsewhere) $L_{\tilde{X}}, L_{\tilde{Y}}, L_k = L_{e_k}$ denote the Lie derivatives along the vector fields $\tilde{X}, \tilde{Y}, e_k$, cf. Sect. A.1.10 (not to be confused with left translations by a group element of a Lie group).
48 (5)	replace $\mathrm{sp}(n)$ by $\mathfrak{sp}(n)$
56 (-10, -7)	Replace the interval $(-\frac{\alpha}{2},\frac{3\alpha}{2})$ by $(t_{\min}+\frac{\alpha}{2}, t_{\max}+\frac{\alpha}{2})$ and $\frac{3\alpha}{2}>t_{\max}$ by $t_{\max}+\frac{\alpha}{2}>t_{\max}$. Then $$\gamma\colon (t_{\min}+\tfrac{\alpha}{2}, t_{\max}+\tfrac{\alpha}{2})\to G,\quad t\mapsto \phi_X(\tfrac{\alpha}{2})\phi_X(t-\tfrac{\alpha}{2})$$ is an integral curve of X with $\gamma(0)=e$. By uniqueness of solutions of ODEs $$\gamma(t)=\phi_X(t) \text{ for all } t\in(t_{\min}+\tfrac{\alpha}{2}, t_{\max}). \qquad (*)$$ If $t_{\min}=-\infty$, then $$\gamma\colon(-\infty, t_{\max}+\tfrac{\alpha}{2})\to G$$ is an extension of $\phi_X\colon(-\infty,t_{\max})\to G$. If $t_{\min}>-\infty$, then $$(t_{\min}, t_{\max}+\tfrac{\alpha}{2})\to G,\quad t\mapsto\begin{cases}\phi_X(t) & t\in(t_{\min},t_{\max})\\ \gamma(t) & t\in(t_{\min}+\frac{\alpha}{2}, t_{\max}+\frac{\alpha}{2})\end{cases}$$ is an extension of $\phi_X\colon(t_{\min},t_{\max})\to G$, well-defined and smooth by eqn. $(*)$.
65 (-4)	replace $\gamma\colon\mathbb{R}\to\mathbb{R}$ by $\gamma\colon\mathbb{R}\to\mathbb{K}$

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69 (-5, -2)	The claim follows without the Taylor formula using $$\tilde{\mu}(0,0)=0, \quad D_{(0,0)}\tilde{\mu}(x,y)=x+y$$ (see p. 70, lines 2 and 6). By the definition of the total derivative $$\lim_{t\to 0}\frac{\|\tilde{\mu}(tx,ty)-\tilde{\mu}(0,0)-D_{(0,0)}\tilde{\mu}(tx,ty)\|}{\|(tx,ty)-(0,0)\|}=0,$$ holds for all $(x,y)\neq(0,0)$. It follows that $$\lim_{t\to 0}\frac{\|\tilde{\mu}(tx,ty)-t(x+y)\|}{t}=0.$$
86 (-12)	For a finite-dimensional real or complex vector space V, the Lie algebra of the Lie group $\mathrm{GL}(V)$ is $\mathfrak{gl}(V)=\mathrm{End}(V)$ as a real vector space (cf. Theorem 1.5.22 and Examples 1.4.3 and 1.4.4).
96 (-8, -3)	unitary representation of a Lie algebra here means skew-Hermitian
115	The direct sums in Theorem 2.4.21 and Corollary 2.4.22 are orthogonal with respect to the Killing form because $B_{\mathfrak{g}}(X,Y)=0$ for all $X\in\mathfrak{z}(\mathfrak{g})$ (hence $\mathrm{ad}_X\equiv 0$) and $Y\in\mathfrak{g}$.
118–119 Theorem 2.5.3	A simpler formulation of Theorem 2.5.3 is: *Let G be a compact simple Lie group. Negative multiples $aB_{\mathfrak{g}}$ of the Killing form, with $a\in\mathbb{R}^-$, are* Ad*-invariant positive definite scalar products on the Lie algebra $\mathfrak{g}$. Every* Ad*-invariant positive definite scalar product on $\mathfrak{g}$ is of this form.*
125 (1-2)	Show that $-F_\lambda$ is a positive definite...
140 (-10)	For $g,h\in G_p$ we calculate
143 (4)	This definition holds for all $p\in M$.
146 (12)	replace $\alpha_{g^{-1}}$ by $c_{g^{-1}}$ (conjugation)
146 (-7)	replace $\alpha_{g^{-1}}$ by $c_{g^{-1}}$ (conjugation)
148 (-8)	replace $\alpha_{g^{-1}}$ by $c_{g^{-1}}$ (conjugation)
204 (5)	*Proof* The subset $E_W=\pi^{-1}(W)$ is an embedded submanifold of E because $\pi\colon E\to M$ is a submersion and $W\subset M$ is an embedded submanifold. Hence the restriction $\pi_M\colon E_W\to W$ is a surjective submersion between smooth manifolds. Let $\{(U_i,\phi_i)\}_{i\in I}$ be a bundle atlas for...
220 (4)	replace $s\colon M\to G$ by $s\colon M\to P$
232 (7)	for a Euclidean bundle metric replace "non-degenerate symmetric" by "Euclidean" (pseudo-Euclidean bundle metrics can be defined analogously)
240 (12)	Since $P\times V\to E$ is a submersion, the map ψ_U is smooth by Lemma 3.7.5.
262 (-14)	We have to verify the conditions defining a connection 1-form. We first check that A is a smooth 1-form: With the inverse of the bundle isomorphism $P\times\mathfrak{g}\overset{\cong}{\to}V$ from 3. in Proposition 5.1.3, the map A is given by the composition $$TP=V\oplus H\xrightarrow{\mathrm{pr}_1}V\xrightarrow{\cong}P\times\mathfrak{g}\xrightarrow{\mathrm{pr}_2}\mathfrak{g}$$ and thus yields a smooth 1-form on P. It is clear that...

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264 (7)	$T_{(z_0,z_1)}S^3 = \{(X_0,X_1) \in \mathbb{C}^2 \mid \mathrm{Re}(\bar{z}_0X_0 + \bar{z}_1X_1) = 0\}$ (the real part of the standard Hermitian scalar product on $\mathbb{C}^n$ is the standard Euclidean scalar product on $\mathbb{R}^{2n}$)
266 (-9)	replace $C^\infty(P,G)^G$ by $\mathcal{C}^\infty(P,G)^G$
266 (-8)	the condition in this definition should hold for all $p \in P$ and $g \in G$; the correct definition is $\mathcal{C}^\infty(P,G)^G = \{\sigma\colon P \to G \text{ smooth} \mid \sigma(p\cdot g) = c_{g^{-1}}(\sigma(p)) = g^{-1}\sigma(p)g \quad \forall p \in P, g \in G\}$.
275 (4)	replace $X_{\sigma(n)}$ by $X_{\sigma(k+l)}$
275 (5)	where $X_1,\ldots,X_{k+l}$ are vector fields on P and the commutators on the right are...
275 (-4)	where X, Y are vector fields on P. We can now state...
287 (-5)	replace $g(0) = e$ by $g(a) = e$
289 (3)	It follows by uniqueness of horizontal lifts that $r_g \circ \gamma_p^*$ is equal to $\gamma_{p\cdot g}^*$.
290 (-9)	there is a closing bracket missing at the end of the formula, i.e. $\ldots(\Phi(\gamma(t))) \in E_x$.
291 (1-5)	Let $q(t)$ be the uniquely determined curve in the fibre P_x such that $$\Pi^A_{\gamma_t}(q(t)) = s(\gamma(t)).$$ The curve $q(t)$ is smooth: By the proof of Theorem 5.8.2, there exists a uniquely determined smooth curve $h(t)$ in G such that $$\Pi^A_{\gamma_t}(s(x)) = \gamma^*(t) = s(\gamma(t))\cdot h(t).$$ Property 4. of Theorem 5.8.4 implies that $q(t) = s(x)\cdot h(t)^{-1}$, in particular $q(t)$ is a smooth curve. We get $$q(t) = s(x)\cdot g(t)$$ with the uniquely determined smooth curve $g(t) = h(t)^{-1}$ in G. Then...
292 (2)	replace $A_s(x)$ by $A_s(X)$
294 (-7) and 295 Fig. 5.2	The Feynman diagram on the left in Fig. 5.2 (cubic interaction vertex does not follow from the Klein-Gordon Lagrangian in Eq. (7.3). The vertex should rather involve one A_μ and two ϕ. The type of interaction in Fig. 5.2 does however appear in the Standard Model, see Fig. 8.14 on page 507.
306 (-9)	replace $d_A\omega$ by $d_\nabla\omega$
319 (7)	by *spinor fields (spinors)* [blank space missing]
319 (10)	the *(orthochronous) spin group* [blank space missing]
320 (16)	also consider *complex bilinear* [blank space missing]
332 Corollary 6.2.12 333 Theorem 6.2.16 334 (11) 335 (1)	replace orthonormal basis $e_1,\ldots,e_n$ by orthogonal basis $e_1,\ldots,e_n$ for (V,Q), i.e. only assume $Q(e_i,e_j) = 0$ for all $i \neq j \in \{1,\ldots,n\}$. The symmetric bilinear form Q is not assumed to be non-degenerate.
333 (-9)	The indices $i_1,\ldots,i_k$ are understood to be pairwise different.
344 (4-9)	Example 6.3.18 can be generalized as follows: Let γ_a, Γ_a be gamma matrices for $\mathrm{Cl}(s,t)$. Then $\gamma_a' = i\gamma_a, \Gamma_a' = i\Gamma_a$ are gamma matrices for $\mathrm{Cl}(t,s)$. More generally, $\gamma_a' = \epsilon_a i\gamma_a, \Gamma_a' = \epsilon_a i\Gamma_a$, where $\epsilon_a \in \{\pm 1\}$ can be chosen for each index a separately, are gamma matrices for $\mathrm{Cl}(t,s)$. If γ_a, Γ_a are gamma matrices for $\mathrm{Cl}(n,0)$, where $n = s+t$, then $\gamma_a' = \epsilon_a\gamma_a, \Gamma_a' = \epsilon_a\Gamma_a$ for $1 \leq a \leq s$ and $\gamma_a' = \epsilon_a i\gamma_a, \Gamma_a' = \epsilon_a i\Gamma_a$ for $s+1 \leq a \leq n$ are gamma matrices for $\mathrm{Cl}(s,t)$.

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347 (under Corollary 6.4.4)	The statement that for n odd $\mathbb{C}\mathrm{l}^0(n) \cong \mathrm{End}(\Delta_n)$ can be identified with the first summand in $\mathbb{C}\mathrm{l}(n) \cong \mathrm{End}(\Delta_n) \oplus \mathrm{End}(\Delta_n)$ is wrong. The subalgebra $\mathbb{C}\mathrm{l}^0(n)$ is diagonal in $\mathbb{C}\mathrm{l}(n)$ (compare with the proof of [88, Chapter I, Prop. 5.12]).
350 (6)	It can be seen that $\mathrm{Spin}^+(s,t) = \{v_1v_2\cdots v_{2r} \mid r = p+q,\ 2p$ of the v_i are in $S^{s,t}_+$ and $2q$ of the v_i in $S^{s,t}_-\}$ (let $v \in S^{s,t}_+, w \in S^{s,t}_-$ and write $w = w_\parallel + w_\perp$ with $w_\parallel = \eta(v,w)v$ and $\eta(v,w_\perp) = 0$; then $wv = vw'$ with $w' = w_\parallel - w_\perp$ and $w' \in S^{s,t}_-$). This expression is more useful for showing the subgroup property and the statement in Definition 6.5.6 for $\mathrm{Spin}^+(s,t)$.
350 (-4)	$\deg^t(u)$ is well-defined because $u^{-1} = v_r^{-1}\cdots v_1^{-1} = (-1)^{\deg^t(u)} v_r \cdots v_1$ (see [13, p. 52-53]) and the transpose map $v_1\cdots v_r \mapsto v_r \cdots v_1$ is well-defined (see [88, Chapter I, eqn. (1.15)]).
352 (3)	replace $\mathrm{R}^{s,t}$ by $\mathbb{R}^{s,t}$
361 (7-8)	...of a G-equivariant left quaternionic vector space on V.
363 Table 6.6	For $\rho = 2, 6 \bmod 8$ there exist also symplectic Majorana spinors (see 4. in Theorem 6.7.20 and [20, Theorem 1.39]).
380 Lemma 6.9.11	In the first line of the proof replace "is a contractible open subset U' of" by "is a contractible embedded subset U' of".
383 (7-10)	(More generally, any metric connection on the tangent bundle, not necessarily torsion-free, defines a unique compatible covariant derivative on S, see [13, Satz 3.2]. Lemma 6.10.3 below still holds in this case, but Lemma 6.10.5 does not necessarily hold.)
389 (1)	...suppose that $\langle\cdot\,,\cdot\rangle_S$ is a Dirac bundle metric...
389 (5)	replace map $\to \mathcal{C}^\infty(M)$ by map $\to \mathbb{C}$
390 (7)	$\epsilon \times s$ is a section over U of the fibre product $\mathrm{Spin}^+(M) \times_M P$ (cf. Remark 6.12.7).
391 (-9)	replace map $\to \mathcal{C}^\infty(M)$ by map $\to \mathbb{C}$
391 (-3)	replace Dirac operator D by D_A
394 (5-6)	For the dimensions of V_+ and V_- see Sect. 8.5.1, in particular Table 8.1 on p. 480 and the discussion following Lemma 8.5.1.
398	In Exercise 6.13.20 the connection on the tangent bundle is assumed to be the torsion-free Levi-Civita connection.
409 (-5)	replace $\mathcal{C}^\infty(M)$ by $\mathcal{C}^\infty(M,\mathbb{K})$
413 (5-6)	the first three terms on the right hand side of the equation should be multiplied with dvol_g
413 (7, 9)	replace $\langle \nabla f, e\rangle_E$ by $\langle e, \nabla f\rangle_E$
414 (-5)	replace $\mathcal{C}^\infty(U,\mathfrak{g})$ by $\mathcal{C}^\infty(U,\mathbb{R})$
415 (6)	the scalar product on the right hand side in the first line is the one defined on p. 414, line -5.
418 (-3)	Theorem 7.2.12 can be applied because by Proposition 5.9.7 the covariant derivative ∇^A on $\mathrm{Ad}(P)$ associated to A is compatible with the bundle metric $\langle\cdot\,,\cdot\rangle_{\mathrm{Ad}(P)}$.
426 (-11, -8, -7) and 427 (2)	replace $\mathbb{C}^r$ by $\mathbb{C}^n$, i.e. identify W with $\mathbb{C}^n$ to avoid confusion with dimension r of the Lie group G.
427 (12)	replace $W = \mathbb{C}^s$ by $W = \mathbb{C}^n$
427 (-9, -8)	replace V by W

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428 Lemma 7.5.8	The claim in Lemma 7.5.8 can also be stated as $$\nabla_X^{f^*A} f^{-1}\Phi = f^{-1}\nabla_X^A \Phi \quad \forall X \in \mathfrak{X}(M).$$ This equation shows that ∇^A is "covariant" with respect to the action of gauge transformations on an associated vector bundle.
428 (9-12)	To prove the claim that $$\langle f\Phi, f\Psi\rangle_E = \langle \Phi, \Psi\rangle_E \quad \forall \Phi, \Psi \in \Gamma(E), f \in \mathcal{G}(P),$$ we consider $x \in M$, $p \in P_x$ and write $\Phi_x = [p, \phi]$, $\Psi_x = [p, \psi]$ with $\phi, \psi \in W$. Then $$f\Phi_x = [f(p), \phi] = [p \cdot \sigma_f(p), \phi] = [p, \rho(\sigma_f(p))\phi]$$ and similarly for $f\Psi_x$. It follows that $$\langle f\Phi_x, f\Psi_x\rangle_E = \langle \rho(\sigma_f(p))\phi, \rho(\sigma_f(p))\psi\rangle_W = \langle \Phi_x, \Psi_x\rangle_E$$ since $\langle \cdot\,, \cdot\rangle_W$ is G-invariant.
439 (3-25)	The Majorana mass term is not identically zero if and only if the Majorana form is symmetric (for commuting spinors) or antisymmetric (for anticommuting spinors), cf. Remark 6.7.7. In particular, for Minkowski spacetime of dimension 4, the Majorana mass term is not identically zero only for anticommuting spinors (this is relevant in Sect. 9.2.5.)
440 (14)	...of U(1) on $\mathbb{C}$ of winding number $k \neq 0$. Suppose that...
469 (1-2)	This means that the basis vectors (α_3, α_4) are obtained by rotating (β_3, β_4) clockwise by the angle θ_W.
479 (3)	replace $V_L =$ by $V_R =$
488 (-15, -13)	We saw above that the Higgs bundle is the vector bundle $$\underline{\mathbb{C}} \otimes E \quad (\cong E), \qquad (8.14)$$ where E is associated to the principal bundle P via a unitary representation on W. Here $\underline{\mathbb{C}}$ denotes...
491 (9-11)	See the grey box at the end of Section 8.5 and the comment for p. 495, lines 3-5 below.
492 (-6, -1) and 494 (-7, -3) item 3.	More details on hypercharge quantization, the compactness of $\mathrm{U}(1)_Y$, the uniqueness of hypercharge assignments, and the mixed gauge-gravitational anomaly can be found in N. Lohitsiri, *Anomalies and the Standard Model of particle physics*, PhD Thesis, University of Cambridge (2020), Sect. 2.2, and references therein.
495 (3-5)	From Table 8.2 on p. 481 and $\mathbb{Q} = \mathbb{T}_3 + \frac{\mathbb{Y}}{2}$ the electric charge of the proton (consisting of two up valence quarks and one down valence quark) is equal to $\frac{1}{2}(1 + 3Y_Q)$ and the electric charge of the electron is equal to $\frac{1}{2}(-1 + Y_L)$, hence the sum is $\frac{1}{2}(3Y_Q + Y_L)$. The third constraint in Table 8.11 shows that this sum is equal to zero.
498 (-9, -8) and 543 Remark 9.2.11 and 552 Remark 9.3.8	For the notion of commuting and anticommuting spinors see Remark 6.7.7. See also the comment for p. 439 lines 3-25 above.
574 (11)	See Remark 2.1.42 for the definition of branching rule.
577 (-9)	The contraction $\lrcorner$ is defined in Definition 6.2.15.

<table>
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<tr><td>587–589</td><td>According to Corollary 8.5.8 the representations of G_{SM} on the fermions in the Standard Model descend to representations of $G_{\mathrm{SM}}/\mathbb{Z}_6$ and hence to representations of $G_{\mathrm{CQ}}/\mathbb{Z}_6$. Since $\mathbb{Q} = \mathbb{T}_3 + \frac{\mathbb{Y}}{2}$, the representations $\mathbb{C}^2 \otimes \mathbb{C}_y$ and $\mathbb{C} \otimes \mathbb{C}_y$ of $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ decompose under restriction to $\mathrm{U}(1)_Q$ into representations of the form $\mathbb{C}_{2q}$, where q is the electric charge and $\beta \in \mathrm{U}(1)_Q$ acts as $z \mapsto \beta^{6q} z$ (cf. Lemma 8.5.1). As the electric charges in the Standard Model are multiples of $\frac{1}{3}$, these representations descend to representations of $\mathrm{U}(1)'_Q$, where $\mathrm{U}(1)_Q \to \mathrm{U}(1)'_Q, \beta \mapsto \beta^2$ is the double cover. It follows that the representations of $G_{\mathrm{CQ}}/\mathbb{Z}_6$ on the fermions descend further to representations of $G'_{\mathrm{CQ}}/\mathbb{Z}_3$, where $G'_{\mathrm{CQ}} = \mathrm{SU}(3)_C \times \mathrm{U}(1)'_Q$ (compare with [104, p. 107]).</td></tr>
<tr><td>593 (-14)</td><td>the notation $\mathcal{N} = 1$ is often used instead of $N = 1$</td></tr>
<tr><td>593 (-11, -10)</td><td>This means that S is a real subspace of minimal dimension of the spinor representation Δ so that the spinor representation of $\mathrm{Spin}^+(V)$ on Δ restricts to S (see the right column of Table 6.6 on p. 363).</td></tr>
<tr><td>597</td><td>In Table 9.2 the formatting for the rows of W-bosons and Winos should be as follows:
<table>
<tr><td rowspan="2">W-bosons</td><td>$W^\pm$</td><td>1</td><td rowspan="2">Winos</td><td>$\tilde{W}^\pm$</td><td>$\frac{1}{2}$</td></tr>
<tr><td>W^0</td><td>1</td><td>$\tilde{W}^0$</td><td>$\frac{1}{2}$</td></tr>
</table></td></tr>
<tr><td>598 (11)</td><td>replace subgroup $\mathrm{SU}(2n)$ by subgroup $\mathrm{SU}(n)$</td></tr>
<tr><td>599 (-15)</td><td>the complex fundamental representation $V = \mathbb{C}^{2n}$ of $\mathrm{SO}(2n)$ decomposes</td></tr>
<tr><td>607 (9-10)</td><td>between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$.</td></tr>
</table>